Alternative Chain Rule Proof

Christian Jones: University of Bath

March 2023

Table of Contents

# Introduction

Here is an alternative proof of the chain rule which uses the - definition of the limit. It’s quite involved, but its a great example if you want more practice with these types of arguments. The typed version here is based off of one presented by Adrian Hill, who previously lectured this course.

# The Chain Rule

Theorem 1 (Chain Rule)

Let and be such that Assume that is differentiable at and is differentiable at . Then the composition is differentiable at with

Proof.

Firstly, note that using the definition of limit, we can recast this problem into the following form: given , we seek such that

Now, fix , and let be determined later. As is differentiable at , such that

Also, as is differentiable at , such that

So, if for

then using (3), we find that

Substituting into (1) then gives for ,

Adding and subtracting yields

Applying the reverse triangle inequality and rearranging,

So, if

then this final inequality implies that for ,

Hence, provided and are respectively defined for and by (1) and (2), and is defined by (4), we find that (\*) is satisfied, and the result follows.

□